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Ising intermittency

Ding-wei Huang

Department of Physics, Chung Yuan Christian University, Chung-li, Taiwan

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Abstract. Recent development of intermittency in the Ising model is reviewed. By means of various realization, the classical spin model is adopted to study the particle number fluctuations and the intermittent behavior. The analytical expressions in one dimension are obtained, both for the models with and without an external field. The onset of intermittency in the Ising model is more likely a characteristic of decoupling into one-dimensional subsystems.

I Introduction

Recently there has been a growing interest in intermittent behavior, both in particle physics and in statistical mechanics [1]. The large fluctuation in multiplicity density observed in high energy collisions seems to indicate that the usual statistical fluctuations are insufficient to describe the phenomena. It is then interesting to compare the fluctuations in statistical models with those shown by experimental data. The Ising model provides the simplest tool for such investigation, and one focus of studies is to determine whether it will show intermittency [2]. From an intuitive point of view, intermittency at the critical point of the Ising model seems natural, since correlations occur on all scales. However, the relationship between the intermittency and phase transition is still unclear. This study is devoted to clarification of the onset of intermittency in the Ising model.

The study of Ising intermittency can be summarized as follows: At first, one has a model of spins, in which each spin S_i takes the values of ± 1. Introducing a transformation between spins S_i and multiplicities n_i, e.g.,

$$n_i = \frac{1}{2}(1 + S_i) \ , \tag{1}$$

a model of particle number fluctuation can be realized in the model of spins, *i.e.,* the model of spins becomes a model of particle production. To study the intermittent behavior, the whole phase space of the multiplicities (or correspondingly the whole lattice of the spins) is further divided into blocks of size L and the block multiplicity k_m is defined as the sum of the multiplicity n_i within the block m. Thus,

$$k_m = \sum_{i \ \in \ B_m} n_i \ , \tag{2}$$

where B_m denotes the block m. One calculates the qth factorial moments F_q defined as

$$F_q(L) = \frac{\langle k_m(k_m-1)\cdots(k_m-q+1)\rangle}{\langle k_m\rangle^q} \ , \tag{3}$$

where $\langle\cdots\rangle$ denotes the average taken first over the blocks and then over the ensemble. Finally, one determines if the factorial moments F_q increase as the size L of the blocks is reduced. If they do, one can say the model shows intermittency. Strictly speaking, intermittency implies more than an increase in factorial moments with a decrease in phase space. A linear behavior on the log-log scale is also implied.

In Sect. II the Ising model without an external field is discussed. In Sect. III the influence of an external magnetic field is discussed. The conclusion is in Sect. IV.

II Ising model without external field

This study begins with the Ising model on a hypercubic lattice with nearest-neighbor interactions. The Hamiltonian is given as

$$\mathcal{H} = -\epsilon \sum_{\langle i,j\rangle} S_i \ S_j \ , \tag{4}$$

where the sum over $\langle i,j\rangle$ is over all nearest neighbors, the spins S_i's take the values of ± 1, and ϵ denotes the interaction strength. Let the linear size of the lattice be R, the total number of the lattice sites $N = R^d$ for the hypercubic lattice in d dimension. There are three parameters in this model: the linear lattice size R, the dimension d, and the product $\beta\epsilon$, where β is the inverse temperature.

To connect with the phenomena of multiparticle production, the N-site lattice is considered to be the overall phase space for the particle production, and a relationship should be established between spin S_i and multiplicity n_i. There is no unique way to do this. In the following subsections, various kinds of realizations for such a relationship are discussed.

A Spin-up as multiplicity

The lattice gas interpretation is usually adopted to provide a connection between spin and multiplicity. Spin-up $S_i = +1$ and spin-down $S_i = -1$ are mapped into the multiplicity $n_i = 1$ and $n_i = 0$, respectively. This can be simply presented by

$$n_i = \frac{1}{2}(1 + S_i) \ . \tag{5}$$

Due to the $Z(2)$ symmetry of the Ising model, the reversed realization of spin-down as particle and spin-up as vacancy does not lead to new physics, which will be shown explicitly later.

Above the phase transition temperature, the average spin $\langle S_i \rangle = 0$ and the total spin $\langle S \rangle = 0$. The corresponding multiplicity $\langle n_i \rangle = \frac{1}{2}$ and the total multiplicity $\langle n \rangle = \frac{N}{2}$ are determined by the total number of sites N. Note that in the above notations the average spin $\langle S_i \rangle$ and $\langle n_i \rangle$ are the ensemble averages taken at the lattice site i.

The multiplicity distribution $P(n)$ is defined as the probability of having n particles in production, or accordingly the probability of having exactly n spins pointing up, regardless their locations on the lattice. The $Z(2)$ symmetry implies that $P(n)$ is symmetrical to the reflection at $n = \langle n \rangle$, *i.e.*,

$$P(n_1) = P(n_2) \ , \quad \text{for } n_1 + n_2 = N = 2\langle n \rangle \ . \tag{6}$$

This can be easily observed, since each configuration is equally probable to that with spin-ups and spin-downs switching into each other; and these two configurations are related by $n_1 + n_2 = N$ which in the symmetric phase equals to $2\langle n \rangle$.

Below the phase transition temperature, the $Z(2)$ symmetry is explicitly broken. The choices of spin-up or spin-down as particles would lead to different phenomena. However, the corresponding physics are easily distinguishable from each other. To choose the direction of total spin as the definition of particle always leads to a total multiplicity larger than $\frac{N}{2}$, while the other choice always leads to a lesser total multiplicity. Around the phase transition point, $\langle S_i \rangle << 1$ and $\langle n \rangle \sim \frac{N}{2}$; this is the case of particular interest. The reflection symmetry of $P(n)$ at $n = \langle n \rangle$ is still a good approximation.

In high energy collisions the total multiplicity increases with an increasing incident energy. In the model this feature is simulated by increasing the lattice sites N with additional energy, which can also be interpreted as the available phase space increasing with increasing energy.

For discussion of the intermittency the N-site lattice is divided into blocks of linear size L, and the number of blocks $M = (R/L)^d$. For each block the block-multiplicity k_m $(m = 1, 2, \cdots, M)$ is defined as

$$k_m = \sum_{i \,\in\, B_m} n_i \ , \tag{7}$$

where B_m denotes the block m. Note that with different realizations the relationships between n_i and S_i may change, while those between k_m and n_i are always kept constant.

Then, one proceeds to study how the factorial moments F_q depend on the block size L (or the number of blocks M). The qth factorial moment is defined as

$$F_q(L) = \frac{\langle k_m(k_m - 1)(k_m - 2) \cdots (k_m - q + 1) \rangle}{\langle k_m \rangle^q} \ . \tag{8}$$

Due to the translational invariance of the Ising model, the ensemble average of any quantity is equivalent for every block. The average over blocks becomes redundant. The correspondence in the multiparticle phenomena is the flatness of the multiplicity density over the phase space, which is valid in the central region of high-energy collisions.

Above the phase-transition temperature, the total multiplicity within a block $\langle k_m \rangle = \frac{1}{2} L^d$. Around the phase-transition temperature, one still has

$$\langle k_m \rangle \sim \frac{1}{2} L^d \gg 1 \ , \ \text{when } L \gg 1 \ . \tag{9}$$

The values of the factorial moments are then similar to those of the standard moments,

$$F_q(L) \sim C_q(L) = \frac{\langle (k_m)^q \rangle}{\langle k_m \rangle^q} \ . \tag{10}$$

As one approaches the phase-transition temperature, the scale invariance of the second-order phase transition implies the following renormalization of the block-spins $\tilde{S}_m$ [3],

$$\frac{1}{L^d} \sum_{i \,\in\, B_m} S_i = \mathcal{Q}(L) \tilde{S}_m \ , \tag{11}$$

where

$$\mathcal{Q}(L) = L^{-(d-2+\eta)/2} \ , \tag{12}$$

independent of the block-spin index m. The renormalized block-spins $\tilde{S}_m$ behave the same as the spins S_i and one has

$$\left\langle \left(\tilde{S}_m \right)^i \right\rangle . = \begin{cases} 0 \ , \text{for odd } i \ , \\ 1 \ , \text{for even } i \ . \end{cases} \tag{13}$$

Then the factorial moments become [4]

$$\begin{aligned} F_q(L) \sim C_q(L) &= \frac{\left\langle \sum_{i=1}^{L^d} (n_i)^q \right\rangle}{\left\langle \sum_{i=1}^{L^d} n_i \right\rangle^q} = \frac{\left\langle \left(1 + \mathcal{Q}(L) \tilde{S}_m \right)^q \right\rangle}{\left\langle \left(1 + \mathcal{Q}(L) \tilde{S}_m \right) \right\rangle^q} \\ &= \frac{\sum_{i=0}^{q} C_i^q \mathcal{Q}^i(L) \left\langle \tilde{S}_m^i \right\rangle}{\sum_{j=0}^{q} C_j^q \mathcal{Q}^j(L) \left\langle \tilde{S}_m \right\rangle^j} \\ &= \sum_{i=0}^{[q/2]} C_{2i}^q \mathcal{Q}^{2i}(L) \ . \end{aligned} \tag{14}$$

Again, the $Z(2)$ symmetry is observed as the above expansion involving only the even power of $\mathcal{Q}(L)$. The other choice of spin-down as particles, $n_i = \frac{1}{2}(1 - S_i)$, leads to the very same results. Explicitly, the first few moments are given as

$$F_2(L) = 1 + \mathcal{Q}^2(L) \ , \tag{15}$$
$$F_3(L) = 1 + 3\mathcal{Q}^2(L) \ , \tag{16}$$
$$F_4(L) = 1 + 6\mathcal{Q}^2(L) + \mathcal{Q}^4(L) \ , \tag{17}$$
$$F_5(L) = 1 + 10\mathcal{Q}^2(L) + 5\mathcal{Q}^4(L) \ . \tag{18}$$

As can be clearly seen, the factorial moments do increase as the block size L decreases. If the small difference between the factorial moments and the standard moments is considered, the factorial moments should be corrected as follows,

$$F_2(L) = 1 + \mathcal{Q}^2(L) - \frac{2}{L^d} \ , \tag{19}$$
$$F_3(L) = 1 + 3\mathcal{Q}^2(L) - \frac{6}{L^d}[1 + \mathcal{Q}^2(L)] + \frac{8}{L^{2d}} \ , \tag{20}$$
$$F_4(L) = 1 + 6\mathcal{Q}^2(L) + \mathcal{Q}^4(L) - \frac{12}{L^d}[1 + 3\mathcal{Q}^2(L)] + \frac{44}{L^{2d}}[1 + \mathcal{Q}^2(L)] - \frac{48}{L^{3d}} \ , \tag{21}$$
$$F_5(L) = 1 + 10\mathcal{Q}^2(L) + 5\mathcal{Q}^4(L) - \frac{20}{L^d}[1 + 6\mathcal{Q}^2(L) + \mathcal{Q}^4(L)] + \frac{140}{L^{2d}}[1 + 3\mathcal{Q}^2(L)] - \frac{400}{L^{3d}}[1 + \mathcal{Q}^2(L)] + \frac{384}{L^{4d}} \ . \tag{22}$$

For $L \gg 1$, such corrections are negligible, as expected. To keep only the first-order terms, one has

$$F_2(L) \sim 1 + \mathcal{Q}^2(L) = 1 + \frac{1}{L^{d-2+\eta}} \ , \tag{23}$$
$$F_3(L) \sim 1 + 3\mathcal{Q}^2(L) = 1 + \frac{3}{L^{d-2+\eta}} \ , \tag{24}$$
$$F_4(L) \sim 1 + 6\mathcal{Q}^2(L) = 1 + \frac{6}{L^{d-2+\eta}} \ , \tag{25}$$
$$F_5(L) \sim 1 + 10\mathcal{Q}^2(L) = 1 + \frac{10}{L^{d-2+\eta}} \ . \tag{26}$$

Obviously, in two dimensions the increase in factorial moments F_q with decreasing L is related to the nonvanishing of the critical exponent η. In other dimensions the factorial moments F_q increase with decreasing L, even with $\eta = 0$. In the cases of the nearest-neighbor Ising model, one has $\eta = 0.25$ for $d = 2$ and $\eta = 0.05$ for $d = 3$. As only the properties of renormalization are utilized to obtain the above results, they are ready to be applied to other spin models with different values of η.

If the increase in F_q with decreasing L were directly related to the critical behavior of the system, one would expect a power-law dependence of F_q on L instead of the formulas shown above. Also notice that the above results are for the parameters $N \to \infty$ and $L >> 1$. For larger values of L, the increase in F_q with decreasing L becomes slower. It can be anticipated that the effect of finite lattice-sites N will enhance the intermittent behavior, which has been shown in [4] numerically, and will be shown analytically in the special case of one dimension to be discussed later.

In [5], another kind of realization has been suggested to restore the seeming brokenness of $Z(2)$ symmetry,

$$n_i = \frac{1}{2}\left[1 + sign(S)\ S_i\right] \ , \tag{27}$$

where $sign(S)$ denotes the sign of the total spin for each configuration. It should be emphasized again that without the external field, the lattice gas interpretation does not break the $Z(2)$ symmetry. The assignment of spin-up or spin-down as particles gives the same results. Within the above realization the total multiplicity for each configuration has a minimum at $\frac{N}{2}$. The resulting multiplicity distribution $P(n)$ vanishes for $n < \frac{N}{2}$, which is inconsistent with the experimental data.

B Analytical results for 1-d model

In the special case of a one-dimensional lattice, analytical results can be obtained. Following the above-mentioned procedures, one obtains the factorial moments F_q analytically as,

$$F_2(L) = 1 - \frac{M}{N} + 2\frac{M}{N}\frac{(a - a^N)}{(1-a)(1+a^N)} - 2\frac{M^2}{N^2}\ \frac{a(1 - a^{\frac{N}{M}})(1 - a^{N-\frac{N}{M}})}{(1-a)^2\ \ (1+a^N)} \ , \tag{28}$$

$$F_3(L) = \left(1 - 3\frac{M}{N} + 2\frac{M^2}{N^2}\right) + \left(6\frac{M}{N} - 12\frac{M^2}{N^2}\right) \times \frac{(a - a^N)}{(1-a)(1+a^N)} - \left(6\frac{M^2}{N^2} - 12\frac{M^3}{N^3}\right) \times \frac{a(1 - a^{\frac{N}{M}})(1 - a^{N-\frac{N}{M}})}{(1-a)^2\ \ (1+a^N)} \ , \tag{29}$$

$$\begin{aligned} F_4(L) = &\left(1 - 6\frac{M}{N} + 11\frac{M^2}{N^2} - 6\frac{M^3}{N^3}\right) \\ &+ \left(12\frac{M}{N} - 72\frac{M^2}{N^2} + 72\frac{M^3}{N^3} - 72\frac{M^3}{N^3}\frac{a}{(1-a)^2}\right) \\ &\times \frac{(a - a^N)}{(1-a)(1+a^N)} - \left(12\frac{M^2}{N^2} - 72\frac{M^3}{N^3} + 72\frac{M^4}{N^4}\right. \\ &\left. - 72\frac{M^4}{N^4}\frac{a}{(1-a)^2}\right)\frac{a(1 - a^{\frac{N}{M}})(1 - a^{N-\frac{N}{M}})}{(1-a)^2\ \ (1+a^N)} \\ &+ \left(12\frac{M^2}{N^2} - 12\frac{M^3}{N^3}\right)\frac{a}{(1-a)^2} \\ &+ 12\frac{M^3}{N^3}\frac{(1+a)(a - a^{\frac{N}{M}})(a + a^{N-\frac{N}{M}})}{(1-a)^3\ \ (1+a^N)} \ , \end{aligned} \tag{30}$$

$$F_5(L) = \left(1 - 10\frac{M}{N} + 35\frac{M^2}{N^2} - 50\frac{M^3}{N^3} + 24\frac{M^4}{N^4}\right) + \left(20\frac{M}{N} - 240\frac{M^2}{N^2} + 760\frac{M^3}{N^3} - 480\frac{M^4}{N^4}\right) \times\frac{(a-a^N)}{(1-a)(1+a^N)} - \left(20\frac{M^2}{N^2} - 240\frac{M^3}{N^3} +760\frac{M^4}{N^4} - 480\frac{M^5}{N^5}\right)\frac{a(1-a^{\frac{N}{M}})(1-a^{N-\frac{N}{M}})}{(1-a)^2\ (1+a^N)} - \left(360\frac{M^3}{N^3} - 1440\frac{M^4}{N^4}\right)\frac{a(a-a^N)}{(1-a)^3\ (1+a^N)} + \left(360\frac{M^4}{N^4} - 1440\frac{M^5}{N^5}\right)\frac{a^2(1-a^{\frac{N}{M}})(1-a^{N-\frac{N}{M}})}{(1-a)^4\ (1+a^N)} + \left(60\frac{M^2}{N^2} - 300\frac{M^3}{N^3} + 240\frac{M^4}{N^4}\right)\frac{a}{(1-a)^2} + \left(60\frac{M^3}{N^3} - 240\frac{M^4}{N^4}\right) \times\frac{(1+a)(a-a^{\frac{N}{M}})(a+a^{N-\frac{N}{M}})}{(1-a)^3\ (1+a^N)}\ , \qquad (31)$$

where $N = R$ the number of lattice sites, $M = R/L = N/L$ the number of blocks, and $a = tanh(\beta\epsilon)$. The three parameters of the model are chosen explicitly as N, M, and a. In the limit of zero temperature $a \to 1$, where the renormalization scheme works, the above results reduce to

$$F_2(L) = 2 - \frac{2}{L}\ , \qquad (32)$$

$$F_3(L) = 4 - \frac{12}{L} + \frac{8}{L^2}\ , \qquad (33)$$

$$F_4(L) = 7 - \frac{48}{L} + \frac{83}{L^2} - \frac{42}{L^3}\ , \qquad (34)$$

$$F_5(L) = 11 - \frac{140}{L} + \frac{535}{L^2} - \frac{670}{L^3} + \frac{264}{L^4}\ , \qquad (35)$$

which are similar to the results of a renormalization scheme substituting $\mathcal{Q}(L) = 1$. As can be seen from these formulas, the factorial moments F_q decrease, instead of increase, with a decreasing L. This absence of intermittency is also expected and is usually related to the absence of phase transition in one dimension. However, it is interesting to note that the intermittency does exist in one dimension as the temperature deviates from zero. The increase in factorial moments F_q with decreasing L can also be shown analytically. In the limit of zero temperature, $N \to \infty$ and $a \to 1$ are taken explicitly and $L = \frac{N}{M}$ is kept finite. If the temperature is set slightly higher, or explicitly $a < 1$ and $a^N \to 0$ in the limiting process, one would obtain the following results instead:

$$F_2(L) = 1 + \frac{1}{L}\left(\frac{3a-1}{1-a}\right) - \frac{1}{L^2}\left(\frac{2a(1-a^L)}{(1-a)^2}\right)\ , \qquad (36)$$

$$F_3(L) = 1 + \frac{3}{L}\left(\frac{3a-1}{1-a}\right) + O\left(\frac{1}{L^2}\right)\ , \qquad (37)$$

$$F_4(L) = 1 + \frac{6}{L}\left(\frac{3a-1}{1-a}\right) + O\left(\frac{1}{L^2}\right)\ , \qquad (38)$$

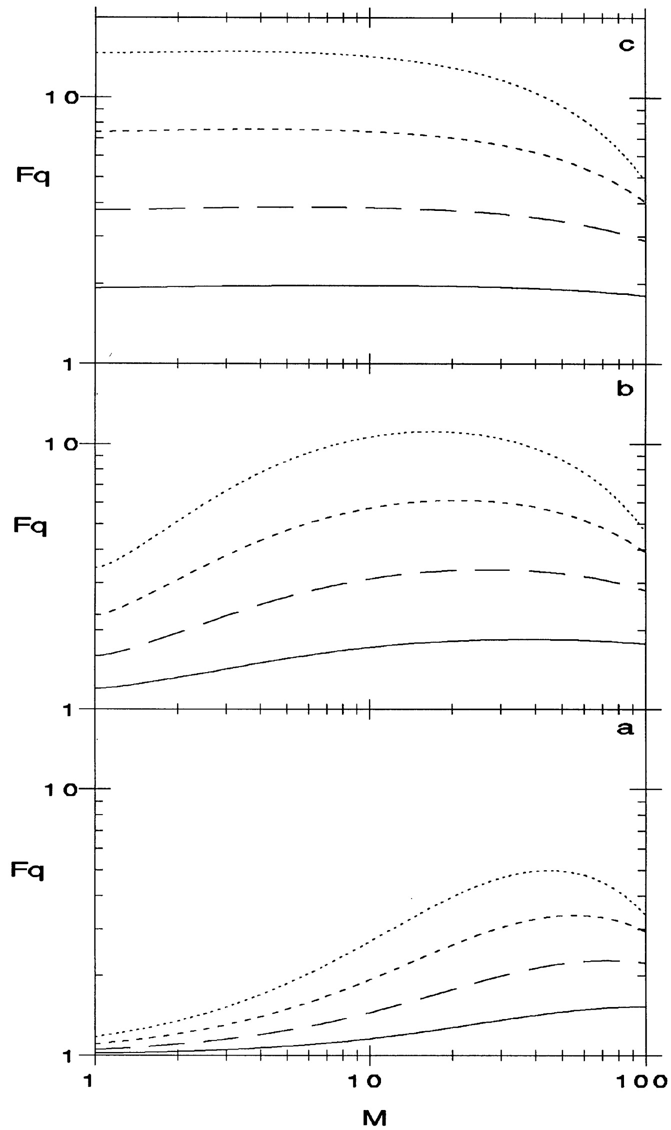

Fig. 1. Factorial moments $F_q(M)$, $q = 2, 3, 4, 5$, of (28-31) on the lattice $N = 1000$ for various values of a: **a** $a = 0.9$, **b** $a = 0.99$, and **c** $a = 0.999$

$$F_5(L) = 1 + \frac{10}{L}\left(\frac{3a-1}{1-a}\right) + O\left(\frac{1}{L^2}\right)\ . \qquad (39)$$

The factorial moments F_q do increase with a decreasing L. It can be simply concluded that the intermittency in this case has nothing to do with the critical behavior, although the obtained formulas are very similar to those given by the renormalization scheme, (23) to (26). In addition, the finite size effect also enhances the intermittency, for which the numerical results are shown in Figs. 1 and 2. Though the $M(=\frac{N}{L})$-dependence of F_q is similar to those observed in experiments, it should be remembered that the total multiplicity $\langle n\rangle = \frac{N}{2}$ is much too large and the multiplicity distribution $P(n)$ has a reflected symmetry at $n = \langle n\rangle$, which is inconsistent with the experimental data.

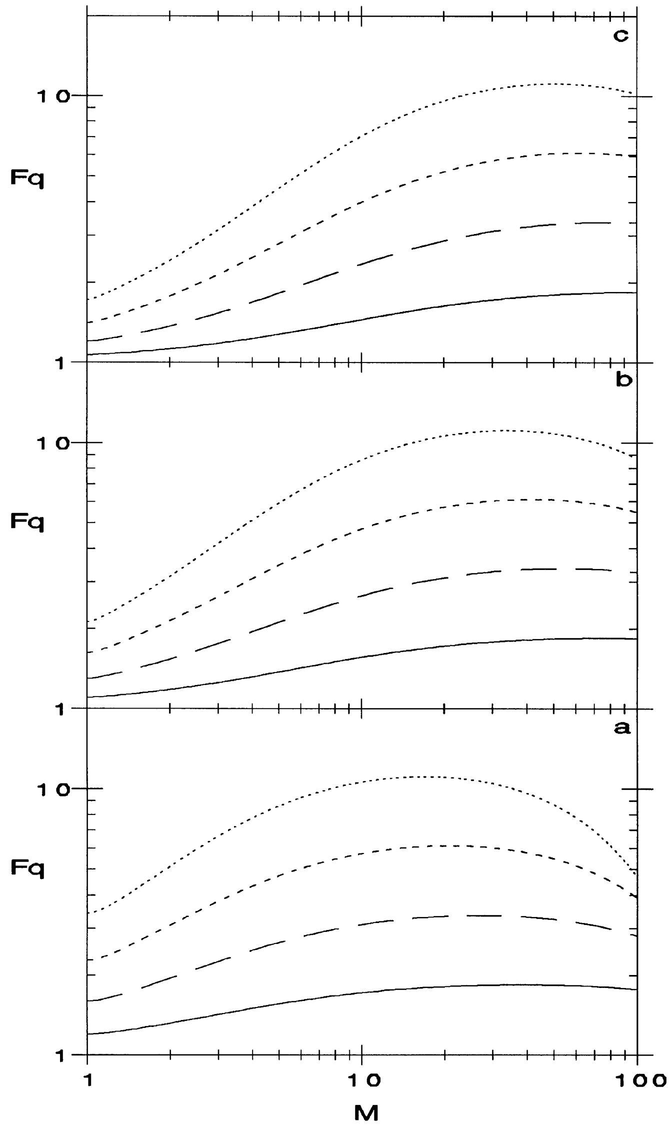

Fig. 2. The same as Fig. 1 for $a = 0.99$ on different sizes of lattices: **a** $N = 1000$, **b** $N = 2000$, and **c** $N = 3000$

C Total-spin as multiplicity

In [6] the naive realization of spin as multiplicity was studied, *i.e.*,

$$n_i = S_i \ . \tag{40}$$

It should be noticed that such multiplicity is not positively definite. The renormalization of the block-spins implies the scale invariance of the standard moments C_q,

$$C_q(L) = \frac{\left\langle \sum_{i=1}^{L^d} (n_i)^q \right\rangle}{\left\langle \sum_{i=1}^{L^d} n_i \right\rangle^q} = \frac{\left\langle \left(\mathcal{Q}(L)\tilde{S}_\alpha \right)^q \right\rangle}{\left\langle \left(\mathcal{Q}(L)\tilde{S}_\alpha \right) \right\rangle^q}$$

$$= \frac{\left\langle \left(\tilde{S}_\alpha \right)^q \right\rangle}{\left\langle \left(\tilde{S}_\alpha \right) \right\rangle^q} = C_q(L = 1) \ . \tag{41}$$

As the total multiplicity $\langle n \rangle$ approaches closer around zero, the differences between standard moments C_q and factorial moments F_q cannot be neglected. The factorial moments become

$$F_2(L) = C_2 - \frac{1}{\langle n(L) \rangle} \ , \tag{42}$$

$$F_3(L) = C_3 - \frac{3C_2}{\langle n(L) \rangle} + \frac{2}{\langle n(L) \rangle^2} \ , \tag{43}$$

$$F_4(L) = C_4 - \frac{6C_3}{\langle n(L) \rangle} + \frac{11C_2}{\langle n(L) \rangle^2} - \frac{6}{\langle n(L) \rangle^3} \ , \tag{44}$$

$$F_5(L) = C_5 - \frac{10C_4}{\langle n(L) \rangle} + \frac{35C_3}{\langle n(L) \rangle^2} - \frac{50C_2}{\langle n(L) \rangle^3} + \frac{24}{\langle n(L) \rangle^4} \ , \tag{45}$$

where $\langle n(L) \rangle = \langle n \rangle \frac{L}{N} = \frac{\langle n \rangle}{M}$ is the block-multiplicity. There is no intermittency in this case.

III Ising model with external field

In high-energy collisions the average number of particles produced is small compared to the maximum number available. For example, in the $\bar{p}p$ collisions at $\sqrt{s} = 540$ GeV, on the average there are only 40 particles produced. This is a small number compared to the maximum number $\sqrt{s}/m_\pi \sim 4000$, considering only the phase space and energy conservation. Within the lattice gas interpretation, introducing an external field is necessary if one wants to simulate the production of a small number of particles on a large lattice. The effect of the external field is to turn the spins away from the direction defined as particles.

With an external magnetic field the Ising Hamiltonian becomes

$$\mathcal{H} = -\epsilon \sum_{\langle i,j \rangle} S_i S_j + h \sum_i S_i \ . \tag{46}$$

There are four parameters in this model: the lattice size N, the dimension d, and the products $\beta\epsilon$ and βh. As the external field breaks the $Z(2)$ symmetry, the direction of spins to be defined as the existence of a particle becomes relevant to the underlying physics. Since one is attempting to simulate the production of a few particles on a large lattice, the appropriate direction to be chosen is obvious. With a positive h in the above Hamiltonian, the multiplicity is still given by

$$n_i = \frac{1}{2}(1 + S_i) \ . \tag{47}$$

A Analytical results for 1-d model

Again, the analytical results can be obtained in one dimension. In this case, there are only three parameters: N,

$\beta\epsilon$, and βh. The two parameters $\beta\epsilon$ and βh are recombined into a and c as follows:

$$a = \frac{cosh(\beta h) - \sqrt{sinh^2(\beta h) + e^{-4\beta\epsilon}}}{cosh(\beta h) + \sqrt{sinh^2(\beta h) + e^{-4\beta\epsilon}}} \ , \tag{48}$$

$$c = \frac{sinh(\beta h)}{\sqrt{sinh^2(\beta h) + e^{-4\beta\epsilon}}} \ . \tag{49}$$

Using parameters N, a, and c, one expresses F_2 explicitly as

$$\begin{aligned} &F_2(M) \\ &= 1 + \frac{4a^N c^2}{[(1-c) + a^N(1+c)]^2} - \frac{2M}{N}\frac{(1+a^N)}{[(1-c)+a^N(1+c)]} \\ &\quad + \frac{M}{N}\frac{(1-c)(1+c)(1+a)(1-a^N)(1+a^N)}{[(1-c)+a^N(1+c)]^2\ (1-a)} - \frac{2M^2}{N^2} \\ &\quad \times \frac{(1-c)(1+c)a(1-a^{\frac{N}{M}})(1-a^{N-\frac{N}{M}})(1+a^N)}{[(1-c)+a^N(1+c)]^2\ (1-a)^2} \ , \end{aligned} \tag{50}$$

where $M = \frac{N}{L}$. Without an external field, $a = tanh(\beta\epsilon)$ and $c = 0$, the above formula reduces to (28) correctly. The analytical results for other F_q moments can also be obtained accordingly. Another two useful parameters, total multiplicity $\langle n \rangle$ and dispersion D, can also be written as

$$\langle n \rangle = \frac{N}{2}\frac{[(1-c)+a^N(1+c)]}{(1+a^N)} \ , \tag{51}$$

$$D \equiv \langle n^2 \rangle - \langle n \rangle^2 \tag{52}$$

$$\begin{aligned} &= N^2 \frac{c^2\ a^N}{(1+a^N)^2} \\ &\quad + \frac{N}{4}(1-c)(1+c)\frac{(1+a)(1-a^N)}{(1-a)(1+a^N)} \ . \end{aligned} \tag{53}$$

The three parameters of the model can be chosen from any one of the sets $(N, \beta\epsilon, \beta h)$, (N, a, c), and $(N, \langle n \rangle, D)$. In discussing the particle production processes, the last set of parameters is preferred, since the total multiplicity $\langle n \rangle$ and dispersion D can be taken from the experimental data directly.

With these analytical expressions, the infinite-lattice limit $N \to \infty$ can be discussed unambiguously. There are two different limits which should be distinguished: the thermodynamic limit and the continuum limit. The thermodynamic limit is well known in statistical mechanics as the limit of infinite lattice sites with the parameters in the Hamiltonian held fixed, in this case $\beta\epsilon$ and βh. The continuum limit is understood as the limit of infinite lattice sites with the physical parameters held fixed, in this case the average multiplicity $\langle n \rangle$ and dispersion D.

In the thermodynamic limit the two parameters $\beta\epsilon$ and βh (or equivalently a and c) are fixed when N approaches infinity. The second factorial moment F_2 can be rewritten as

$$F_2(M) = 1 - \frac{M}{N}\frac{2}{(1-c)} + \frac{M}{N}\frac{(1+c)(1+a)}{(1-c)(1-a)}$$

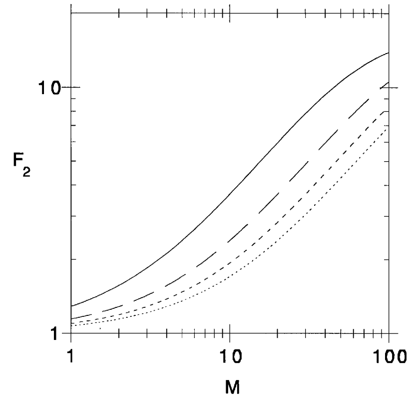

Fig. 3. Factorial moment $F_2(M)$ of (50) with fixed $a = 0.86$ and $c = 0.92$ on different sizes of lattices: the *solid line* for $N = 1000$, the *long-dashed line* for $N = 3000$, the *dashed line* for $N = 3000$, and the *dotted line* for $N = 4000$

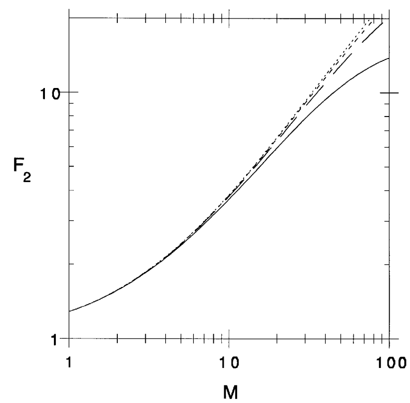

Fig. 4. The same as Fig. 3 with fixed $\langle n \rangle = 40$ and $D = 500$

$$-\frac{2M^2}{N^2}\frac{(1+c)}{(1-c)}\frac{a(1-a^{\frac{N}{M}})}{(1-a)^2} + O\left(a^N\right) \ . \tag{54}$$

It can be seen from the formula that in the limit $a^N \to 0$, the factorial moments F_q scale with $\frac{M}{N}$. If one plots F_q vs. M as the experimental data shows, the increase in F_q with an increasing M will be slower as N becomes larger. The numerical results for different N are shown in Fig. 3. From this scaled property it can be simply concluded that the intermittency does not survive in the thermodynamic limit. Also it should be noted that in this limit the total multiplicity $\langle n \rangle$ diverges as N and the dispersion D as N^2.

In the continuum limit the two parameters $\langle n \rangle$ and D are fixed in the limit $N \to \infty$. The two parameters a and c are changing accordingly as N increases. Numerical results for various values of N are shown in Fig. 4, to be contrasted with Fig. 3. In this case the intermittency not only survives but enhances, which can also be shown explicitly. In the limit $N \to \infty$, one has

$$a \to \frac{D - \langle n \rangle}{D + \langle n \rangle} \ \text{ and } \ c \to 1 \ . \tag{55}$$

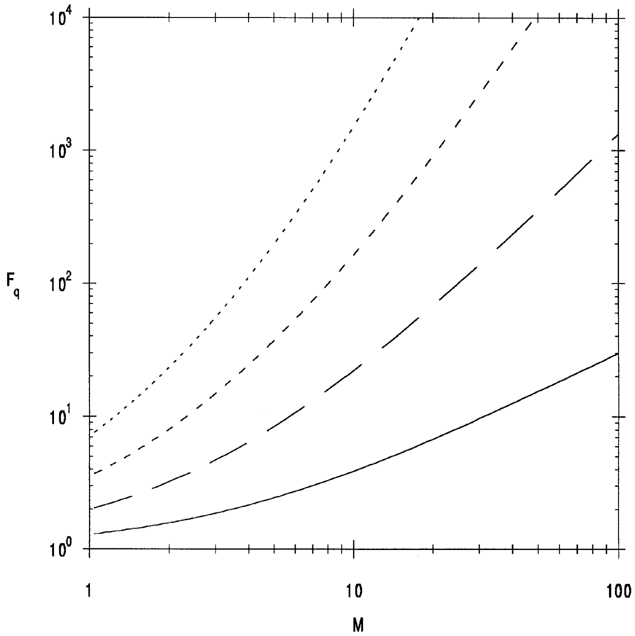

Fig. 5. Factorial moment $F_q(M)$, $q = 2, 3, 4, 5$, of (56–59) in the continuum limit for $\langle n \rangle = 40$ and $D = 500$

In this case, (28) can also be explicitly written as

$$F_2(M) = 1 + M \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right) . \quad (56)$$

The first few moments F_q are given as [7], see Fig. 5,

$$F_3(M)=1 + 3M \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right) + \frac{3}{2} M^2 \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^2 , \quad (57)$$

$$F_4(M)=1 + 6M \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right) + 9M^2 \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^2 + 3M^3 \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^3 , \quad (58)$$

$$F_5(M)=1 + 10M \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right) + 30M^2 \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^2 + 30M^3 \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^3 + \frac{15}{2} M^4 \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^4 . \quad (59)$$

In general, all the factorial moments can be written explicitly as

$$F_q(M)=1 + \sum_{i=1}^{q-1} M^i \frac{(q-1)!\, q!}{2^i\, i!\, (q-i-1)!\, (q-i)!} \times \left(\frac{D - \langle n \rangle}{\langle n \rangle^2} \right)^i . \quad (60)$$

It is also noted that the multiplicity distributions $P(n)$ thus obtained are very similar to the widely used negative binomial distributions [7].

It should be noted that in the case of a one-dimensional lattice there are three scales: N, L, and M, related by $N = M \cdot L$. In this section the limits $N \to \infty$ and $L \to \infty$ are taken, with M kept finite; while in the previous section the appropriate limits were taken by $N \to \infty$ and $M \to \infty$, with L kept finite. Since the formulas in both sections are obtained analytically, they are exact for any values of N, L, and M and can be used to demonstrate these two kinds of limits unambiguously.

B In higher dimensions

The above results are ready to be extended to higher dimensions $d \geq 2$. The intermittency vanishes in the thermodynamic limit and survives in the continuum limit. In the latter the factorial moments are given by the same formulas, (60), in one dimension. The simple reasoning is as follows. To keep the physical parameters $\langle n \rangle$ and D fixed at finite values in the limit $N \to \infty$, the system must be decoupled into one-dimensional subsystems.

To keep the parameter $\langle n \rangle$ finite in the infinite-N limit, most of the spins are pointed in the down direction. There is only a finite number of spin-ups on an infinite lattice. Even with the help of an external field, the system has to be at zero temperature, $T = 0$, to have a negligible fraction of spin-ups.

To keep the parameter D finite in the limit $N \to \infty$, the system has to be at the critical temperature $T = T_c$. In the multiparticle production of high-energy collisions, the large fluctuations are observed as $D \propto \langle n \rangle^2$, in contrast to the conventional statistical fluctuations $D \propto \langle n \rangle$. In the limit $N \to \infty$, such large fluctuations imply that the corresponding spin system has nonvanishing long-range correlations, *i.e.,* the critical behavior is expected.

With these two requirements, the system must be $T = T_c = 0$. If one begins with an anisotropic Ising model in a higher dimension $d \geq 2$, such requirements imply that all of the interactions must vanish except for those in one direction. The lattice is then decoupled into one-dimensional sublattices. The factorial moments $F_q(M)$ can be easily obtained from convolution. The analytical expression is the same as in the case of one dimension, (60).

IV Discussion and conclusion

The Ising model provides a simple tool to study the fluctuations in multiparticle production. With a lattice-gas interpretation, the model does show fluctuations similar to what has been observed in experimental data. In high-energy collisions only a few particles are produced over a large available phase space. The dispersion is large. The factorial moments increase with decreasing block size. To reproduce these desired features from the Ising model, the parameters of the Hamiltonian should be chosen appropriately. The ferromagnetic interaction and the external field are favorable to evoke such features in the simulation.

Given the analytical results in one dimension, many puzzling situations can be clarified. On the infinite lattice the intermittency survives in the continuum limit but vanishes in the thermodynamic limit, while the finite size effect does make the factorial moments increase with a decreasing block size up to a certain level.

In higher dimensions the intermittency exists only in the limit where the system can be decoupled into one-dimensional chains. The finite size effect will enhance the intermittent behavior, as shown both from the renormalization scheme and the numerical work. One may simply conclude that, discussed in this paper, onset of intermittency in the Ising model is not directly related to the phase transition, but more likely to the characteristic of decoupling into one-dimensional subsystems.

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